APPENDIX

This appendix presents the definitions and elementary properties of graphs, digraphs and partial orders used in the book. Since we anticipate that most of these will be familiar to most readers, we have organized the appendix to facilitate its use as a reference.

A.1 GRAPHS

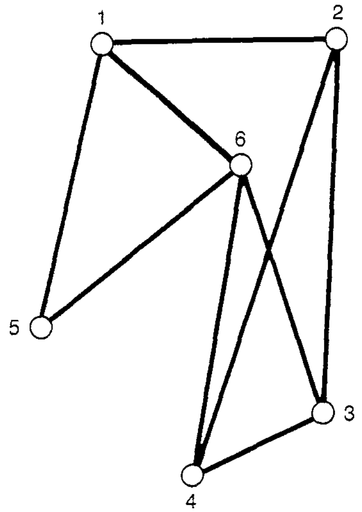
An *undirected graph* $G = (N, E)$ consists of a set N of elements called *nodes* and a set E of unordered pairs of nodes called *edges*. Diagramatically a graph is usually drawn with its nodes as points and its edges as line segments connecting the corresponding pair of nodes, as in Fig. A-1.

A *path* in a graph $G = (N, E)$ is a sequence of nodes $v_1, v_2, \ldots, v_k$ such that $[v_i, v_{i+1}] \in E$ for $1 \leq i < k$. Such a path is said to connect v_1 and v_k. For example, 1, 2, 4, 3, 6, 5 is a path connecting 1 and 5 in Fig. A-1.

A graph $G = (N, E)$ is *connected* if there is a path connecting every pair of nodes. The graph in Fig. A-1 is connected. Fig. A-2 illustrates an unconnected graph.

A graph $G' = (N', E')$ is a *subgraph* of G if $N' \subseteq N$ and $E' \subseteq E$. For example, the graph of Fig. A–2 is a subgraph of that in Fig. A–1.

A *partition* of G is a collection $G_1 = (N_1, E_1), \ldots, G_k = (N_k, E_k)$ of subgraphs of G such that each G_i is connected and the node sets of these subgraphs are pairwise disjoint; i.e., $N_i \cap N_j = \{\}$ for $1 \leq i \neq j \leq k$. (We use "{ }" to denote the empty set.) Each G_i in the collection is called a *component* of the partition. In this definition we do *not* require $\cup_{i=1}^{k} N_i = N$ or $\cup_{i=1}^{n}$



$N = \{1,2,3,4,5,6\}$

$E = \{\{1,2\}, \{1,5\}, \{1,6\}, \{2,3\}, \{2,4\}, \{3,4\}, \{3,6\}, \{4,6\}, \{5,6\}\}$

FIGURE A–1
A Graph $G = (N, E)$

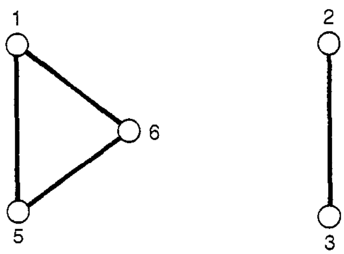
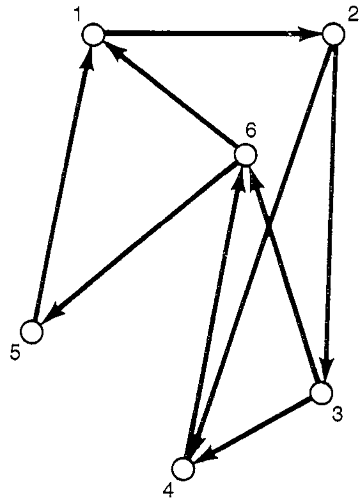


FIGURE A–2
Unconnected Graph

$E_i = E$ (but $\cup_{i=1}^{n} N_i \subseteq N$ and $\cup_{i=1}^{n} E_i \subseteq E$). Thus the graph in Fig. A–2 is a partition of the graph in Fig. A–1, consisting of two components.

A.2 DIRECTED GRAPHS

A *directed graph* (or *digraph*) $G = (N, E)$ consists of a set N of elements called *nodes* and a set E of ordered pairs of nodes, called *edges*. Diagrammatically a digraph is drawn with points representing the nodes and an arrow from point *a*



$N = \{1, 2, 3, 4, 5, 6\}$

$E = \{(1, 2), (2, 3), (2, 4), (3, 4), (3, 6), (4, 6), (5, 1), (6, 1), (6, 5)\}$

FIGURE A–3
A Digraph

to point b representing the edge (a, b). An example of a digraph is shown in Fig. A–3.

A *path* in a digraph G is a sequence of nodes $n_1, n_2, \ldots, n_k$ such that $(n_i, n_{i+1}) \in E$ for $1 \le i < k$. Such a path is said to be *from* n_1 *to* n_k. By convention a single node n constitutes a *trivial path*. (This is different from an edge from n to itself, which is a path from n to n.) A path is *simple* if all nodes, except possibly the first and last in the sequence, are distinct. A *cycle* is a simple nontrivial path where the first and last nodes are identical. For example, 1, 2, 3, 4, 6, 5, 1 is a cycle in the digraph of Fig. A–3. If $n_1, n_2, \ldots, n_k, n_1$ is a cycle, then each edge (n_i, n_{i+1}) for $1 \le i < k$ and (n_k, n_1) is said to be *a member of* or *in* the cycle. A cycle is *minimal* if for every two nodes n_i and n_j in the cycle, if $(n_i, n_j) \in E$, then (n_i, n_j) is in the cycle. The previous example is not a minimal cycle because 1, 2, 3, 6, 1 is a subsequence of it that is also a cycle (indeed a minimal one).

A.3 DIRECTED ACYCLIC GRAPHS

A *directed acyclic graph* or *dag* is a digraph that contains no cycles. Examples of dags are shown in Fig. A–4 and Fig. A–5. A *source* is a node with no incoming edges. A dag must necessarily contain at least one source (why?). A dag with a unique source is called a *rooted dag* and the source is called the dag's *root*. The dag in Fig. A–4 is rooted while that in Fig. A–5 is not.

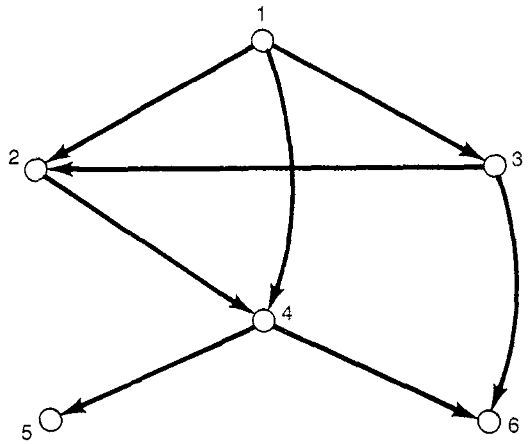


FIGURE A–4
A Rooted Dag

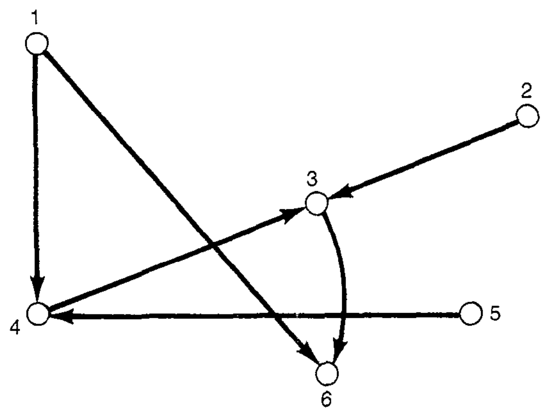


FIGURE A–5
A Dag With Many Sources

If (a, b) is an edge in a dag, a is called a *parent* of b and b is a *child* of a. If there is a path from a to b, then a is an *ancestor* of b and b a *descendant* of a. a is a *proper ancestor* of b if it is an ancestor of b and $a \neq b$; "*proper descendant*" is defined analogously. Note that the acyclicity of a dag implies that a node cannot be both a proper ancestor and a proper descendant of another. If

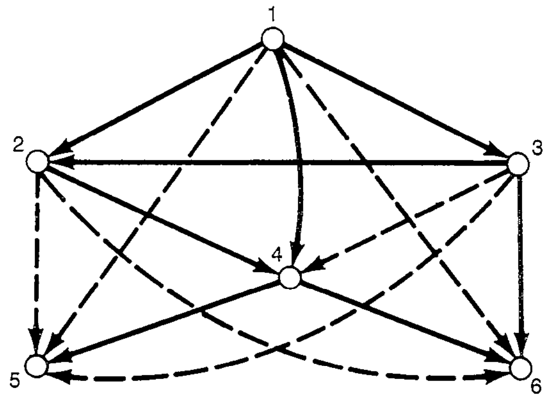


FIGURE A–6
Transitive Closure of a Rooted Dag

a and b are distinct nodes neither of which is an ancestor of the other, we say that a is *unrelated* to b.

A *topological sort* of a digraph G is a sequence of (all) the nodes of G such that if a appears before b in the sequence, there is no path from b to a in G. A fundamental characterization of dags is provided by Proposition A.1.

Proposition A.1: A digraph can be topologically sorted iff it is a dag. □

For a given dag, there may exist several topological sorts. For example, the dag in Fig. A–4 has two topological sorts, namely,

1, 3, 2, 4, 5, 6 and 1, 3, 2, 4, 6, 5.

The *transitive closure* of a digraph $G = (N, E)$ is a digraph $G^+ = (N, E^+)$ such that $(a, b) \in E^+$ iff there is a non-trivial path from a to b in G. Informally, G^+ has an edge anywhere G has a (non-trivial) path. It is easy to show that

Proposition A.2: G^+ is a dag iff G is a dag. □

A more "procedural" definition of transitive closure of a digraph G is:

```
begin
  G+ := G;
  while G+ contains edges (a, b) and (b, c) but not (a, c) for some a, b, c
      in N do add edge (a, c) to G+
end
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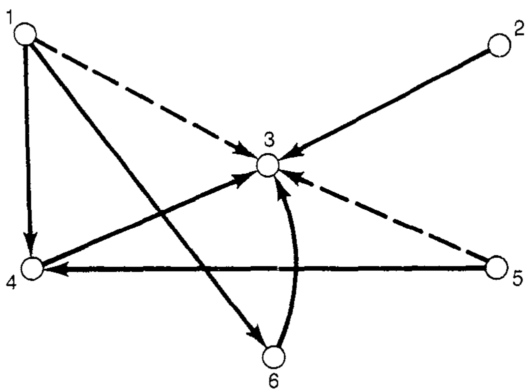

FIGURE A–7
Transitive Closure of a Dag with Many Sources

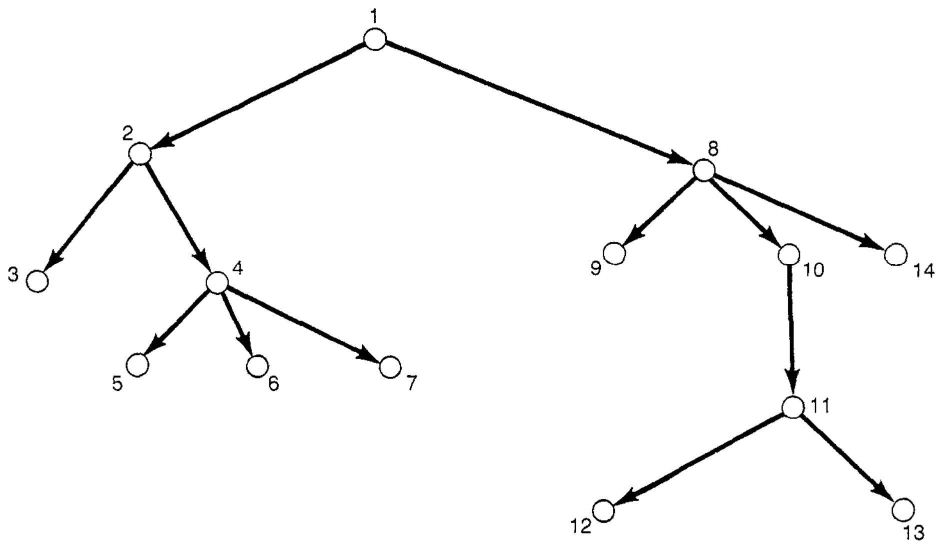

FIGURE A–8
A Tree

The transitive closures of the dags in Fig. A–4 and Fig. A–5 are shown, respectively, in Fig. A–6 and Fig. A–7. The added edges are drawn in broken lines. A digraph is *transitively closed* if it is equal to its own transitive closure (i.e., $G = G^+$). Obviously, the transitive closure of any graph is transitively closed.

A *tree* is a rooted dag with the additional property that there is a unique path from the root to each node. Figure A–8 shows a tree. Ordinarily tree edges are drawn without arrows; the implicit direction is "away from the root." Since there is a unique path from the root to each node, this convention is unambiguous.

Also, the uniqueness of the path from the root to a node implies that in a tree each node has a *unique* parent, except for the root, which has no parent at all. A node may have several children, however.

A.4 PARTIAL ORDERS

A *partial order* $L = (\Sigma, <)$ consists of a set Σ called the *domain* of the partial order and an irreflexive, transitive binary relation $<$ on Σ.[1]

If $a < b$ we say that *a precedes b* and that *b follows a* in the partial order. If neither of two distinct elements precedes the other, the two elements are *incomparable* in the partial order.

A partial order $L' = (\Sigma', <')$ is a *restriction* of L on domain Σ' if $\Sigma' \subseteq \Sigma$ and for all $a, b \in \Sigma'$, $a <' b$ iff $a < b$. L' is a *prefix* of L, written $L' \leq L$, if L' is a restriction of L and for each $a \in \Sigma'$, all predecessors of a in L (i.e., all elements $b \in \Sigma$ such that $b < a$) are also in Σ'.

A partial order $L = (\Sigma, <)$ can be naturally viewed as a dag $G = (N, E)$ where $N = \Sigma$ and $(a, b) \in E$ iff $a < b$. That G is acyclic follows from the irreflexivity and transitivity of $<$. To see this, suppose G had a cycle $a_1, a_2, \ldots, a_k, a_1$. By construction of G, $a_1 < a_2, a_2 < a_3, \ldots, a_k < a_1$; by transitivity of $<$, $a_1 < a_1$, contradicting the irreflexivity of $<$. Moreover, the transitivity of $<$ implies that G is a transitively closed graph.

Conversely, we can construct a partial order $L = (\Sigma, <)$ from a given dag $G = (N, E)$ by taking $\Sigma = N$ and $a < b$ iff $(a, b) \in E^+$, where $G^+ = (N, E^+)$ is the transitive closure of G. To verify that L is indeed a partial order, note that $<$ is irreflexive (because $(a, a) \notin E^+$ for any $a \in N$, since G is acyclic), and $<$ is transitive (because G^+ is transitivity closed).

Thus we can regard a partial order as a dag and vice versa.

[1] A binary relation $<$ on Σ *is irreflexive* if, for all $a \in \Sigma$, $a \not< a$ (i.e., "$a < a$" is false); it is *transitive* if, for all $a, b, c \in \Sigma$, and $a < b$ and $b < c$ imply $a < c$.